# Three-dimensional model versus upscaled mixed sharp-diffuse models for saltwater intrusion. Numerical results.

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MAMERN-VI Conference

1-5 June 2015 - Pau

# Outline

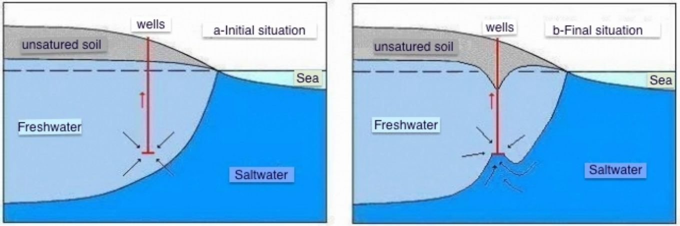


FIGURE : Aquifer without depression - Aquifer with depression

There exist hydraulic exchanges between fresh groundwater and seawater.
They are slow in "natural conditions" and thus are often forgotten and replaced by a quasi-equilibrium between two fluid layers (**Ghyben–Herzberg approximation**).

**But... in case of more drastic conditions (human activities, climatic events)**

the static picture fails : we have to obtain efficient and accurate models to simulate the displacement of **fresh and salt water 'fronts'** in coastal aquifer for the optimal exploitation of groundwater.

1. **Physically based model**

   Fresh- and saltwater are miscible fluids, the aquifer is an unsaturated medium :
   - degenerate equations ;
   - no interfaces but diffuse fronts.

   The approach is thus very heavy from theoretical and numerical points of view.

2. **Hidden sharp interfaces**

   First level of approximation : Fresh- and saltwater are assumed immiscible.

   The explicit tracking of interfaces remains unworkable to implement without further assumptions...

3. **Sharp interfaces**

   Next level of approximation : The two fluids are immiscible and no mass transfert occurs between the fresh and the salty area.

   **Advantage :** This approach gives informations concerning the displacement of the saltwater front.

   **Disadvantage :** It doesn't describe neither take into account the behavior of the real transition zone.

We propose a **mixed sharp-diffuse interfaces approach** for conciliating

- the (numerical) simplicity of a sharp interface approach,
- the actual existence of two transition areas (salinization & desaturation).

**Main points**

- The price to pay for the sharp strategy is the **analysis of two free boundaries**.
- It is compensated by an **upscaling procedure** :

  ⊕ reducing the dimension of the problem from 3D to 2D ;

  ⊕ re-allowing the existence of mass transferts between the areas thanks to dynamic boundary conditions.
- We **superimpose on the interfaces a phase-field model** for re-including the existence of diffuse transitions.

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**Coupling the phase-field approach with the conservation principles**

- The 3D phase-field model is coupled to the 2D upscaled model through the fluxes characterization.
- We use a **tri-stable** phase-field model : **existence of the 'virtual' sharp interface is recovered by an implicit function analysis.**

**Step 1.** The derivation of the 3D model is based on the coupling of

- the mass conservation principle : $\partial_t(\phi\rho) + \nabla\cdot(\rho q) = \rho Q,$
- the Darcy's law : $q = -K\nabla\Phi - \frac{k}{\mu}(\rho-\rho_0)g\nabla z,$

written for freshwater and saltwater.

▶ 2 equations, 2 'functional' unknowns and 2 domain unknowns...

**Step 2.** Vertical upscaling :

$$S_f B_f \partial_t \tilde{\Phi}_f = \nabla'\cdot(B_f \tilde{K}_f' \nabla' \tilde{\Phi}_f) - q_{f|z=h_1^-}\cdot\nabla(z-h_1^-) + q_{f|z=h^-}\cdot\nabla(z-h^-) + B_f\tilde{Q}_f, \quad (1)$$

$$S_s B_s \partial_t \tilde{\Phi}_s = \nabla'\cdot(B_s \tilde{K}_s' \nabla' \tilde{\Phi}_s) + q_{s|z=h_2}\cdot\nabla(z-h_2) - q_{s|z=h^-}\cdot\nabla(z-h^-) + B_s\tilde{Q}_s, \quad (2)$$

▶ 2 equations, 4 unknowns...

**Step 3.** We use the equilibrium of the aquifer pressure with atmospheric conditions for eliminating 2 of the 4 unknowns.

**Step 4. Fluxes across the interfaces :**

This part is fundamental because we re-include now existence of miscible zones,

- For instance between fresh and salt water :

We consider a tri-stable Allen-Cahn type equation :

$$\partial_t F_h + \vec{v}\cdot\nabla F_h - \delta_h \Delta' F_h + \frac{2F_h(F_h - c_s/2)(F_h - c_s)(3F_h^2 - c_s^2/4)}{\delta_h} = 0.$$

Set $\{(x_1,x_2,z)\, s.t.\, F_h(x_1,x_2,z,t) = c_s/2\}$ represents the sharp interface at time $t$. Using an implicit function result and projections of the fluxes, we obtain

$$q_{f|z=h^-}\cdot\nabla(z-h^-) = \phi\big(\mathcal{X}_0(-h)\partial_t h - \delta_h \nabla'\cdot(\mathcal{X}_0(-h)\nabla' h)\big).$$

The latter relation is a regularized Stefan type boundary condition.
This corresponds with our philosophy of diffuse interface tracking.

Setting $T_s(u) = h_2 - u$ (for the salty width), we get :

$$
(\mathcal{M}) \quad \begin{cases}
\phi\partial_t h - \nabla' \cdot \big(K\,T_s(h)\nabla' h\big) - \nabla' \cdot \big(\delta\phi\nabla' h\big) \\
\qquad - \nabla' \cdot \big(K\,T_s(h)\nabla' h_1\big) - \phi q_{Ls}(x, h_1, h) = -\tilde{Q}_s\,T_s(h), \\
\phi\partial_t h_1 - \nabla' \cdot \big(K\mathcal{X}_0(h_1)\big((h - h_1) + T_s(h)\big)\nabla' h_1\big) \\
\qquad - \nabla' \cdot \big(\delta\phi\nabla' h_1\big) - \nabla' \cdot \big(K\,T_s(h)\mathcal{X}_0(h_1)\nabla' h\big) \\
\qquad - \phi q_{Lf}(x, h_1, h) - \phi q_{Ls}(x, h_1, h) = -\tilde{Q}_f(h - h_1) - \tilde{Q}_s\,T_s(h).
\end{cases}
$$

$$
h = h_D, \quad h_1 = h_{1,D} \quad \text{in } \Gamma \times (0, T), \tag{3}
$$

$$
h(0, x) = h_0(x), \quad h_1(0, x) = h_{1,0}(x) \quad \text{in } \Omega. \tag{4}
$$

We consider an open bounded domain $\Omega$ of $\mathbb{R}^2$. Its boundary $\Gamma$ is assumed $\mathcal{C}^1$.

The time interval of interest is $(0, T)$, $T > 0$ and we set

$$
W(0, T) := \left\{\omega \in L^2(0, T; V),\ \partial_t\omega \in L^2(0, T; V')\right\}.
$$

**Theorem :**

Assume a low spatial heterogeneity for the hydraulic conductivity tensor :

$$K_- \leq K_+ \leq 2K_-.$$

Then for any $T > 0$, problem $(\mathcal{M})$, (3)–(4) admits a weak solution $(h, h_1)$ satisfying

1. $(h - h_D, h_1 - h_{1,D}) \in W(0,T) \times W(0,T)$.
2. $0 \geq h_1(t,x) \geq h(t,x) \geq h_2$ for a.e. $x \in \Omega$ and for any $t \in (0,T)$.

**Strategy of the proof**

We introduce a **penalization based on the propagation speed of the salty front**... requiring *a priori* compactness results in $H^1$.

**[1]** [Choquet, **Diedhiou**, Rosier]. *Mathematical analysis of a sharp-diffuse interfaces model for seawater intrusion*. To appear un JDE.

We consider the displacement of two miscible species transported by a compressible flow in a porous deformable medium. This model is based on the coupling of

- the mass conservation principle for the first specie :

$$\partial_t(\rho_1 \phi c) + \operatorname{div}(\rho_1 c q) - \operatorname{div}(\rho_1 \phi \mathcal{D}(q) \nabla c) = \rho_1 (q_i - c q_s),$$

- the mass conservation principle for the second specie :

$$\partial_t(\rho_2 \phi (1-c)) + \operatorname{div}(\rho_2 (1-c) u) - \operatorname{div}(\rho_2 \phi \mathcal{D}(q) \nabla (1-c)) = -\rho_2 (1-c) q_s,$$

- the Darcy's law :

$$q = -\frac{k}{\mu}(\nabla p + \rho g \nabla z).$$

We assume

- the slight compressibility of the two miscible species and the rock,
- the variation of density in the direction of the flow is very small (Bear-Assumption).

We get the 3D Model for transport in $\Omega \times [-10, 0] \in \mathbb{R}^3$ :

$$(\mathcal{N}) \quad \begin{cases} \partial_t \theta(p) + \theta(p) a(c) \partial_t p + \nabla \cdot u = q_i - q_s, \qquad u = -\kappa(\theta(p)) \nabla p, \\ \theta(p) \partial_t c + \theta(p) b(c) \partial_t p + u \cdot \nabla c - \nabla \cdot (\theta(p) \mathcal{D}(u) \nabla c) = q_i (1 - c) \end{cases}$$

$$\frac{\partial p}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \quad \text{in } \Gamma^a \times (0, T), \tag{5}$$

$$p(0, x) = p_0(x), \quad c(0, x) = c_0(x) \quad \text{in } \Omega \times [0, 5]. \tag{6}$$

where we set

$$a(c) = (z_1 - z_2) c + z_2, \quad b(c) = (z_1 - z_2) c (1 - c),$$

and $\Gamma^a$ is the boundary of $\Omega \times [-10, 0]$.

In the following, we will compare the interface between freshwater and salt water against the 2D model and 3D model.
Today we assume that :

1. the porous medium is saturated and incompressible, then $\theta = \phi$ is constant and the interface $h_1$ between saturated and unsaturated medium is fixed,
2. the compressibility of the two fluids is the same, then ($z_1 = z_2$ ) and $a(c) = z_2$ $b(c) = 0$.

$$(\mathcal{M}) \quad \begin{cases} \phi\partial_t h - \nabla' \cdot \big(K\, T_s(h)\nabla' h\big) - \nabla' \cdot \big(\delta\phi\nabla' h\big) \\ \qquad\qquad - \nabla' \cdot \big(K\, T_s(h)\nabla' h_1\big) = -\tilde{Q}_s\, T_s(h), \\ -\nabla' \cdot \big(K\big((h - h_1) + T_s(h)\big)\nabla' h_1\big) - \nabla' \cdot \big(\delta\phi\nabla' h_1\big) \\ \qquad\qquad - \nabla' \cdot \big(K\, T_s(h)\nabla' h\big) = -\tilde{Q}_f(h - h_1) - \tilde{Q}_s\, T_s(h). \end{cases}$$

$$\frac{\partial h_1}{\partial n} = 0, \quad \frac{\partial h}{\partial n} = 0 \quad \text{in } \Gamma \times (0, T),$$

$$h(0, x) = h_0(x), \quad h_1(0, x) = h_{1,0}(x) \quad \text{in } \Omega.$$

We use a semi-implicit Euler method in time and FEM in space.

We use a mixed finite element method for solving $(p, u)$ and $(c, u_c)$

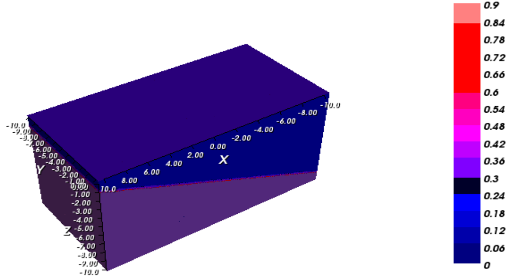
$$
(\mathcal{N}) \quad \begin{cases}
\phi z_2 \partial_t p + \nabla \cdot u = q_i - q_s, \\
u = -\kappa(\phi)\nabla p, \\
\phi \partial_t c + c\nabla \cdot u - \nabla \cdot u_c = q_i(1-c), \\
u_c = -\phi \mathcal{D}(u)\nabla c + uc,
\end{cases}
$$

$$
u \cdot n = 0, \quad u_c \cdot n = 0 \quad \text{in } \Gamma^a \times (0, T),
$$

$$
p(0,x) = p_0(x), \quad c(0,x) = c_0(x) \quad \text{in } \Omega \times [0,5].
$$

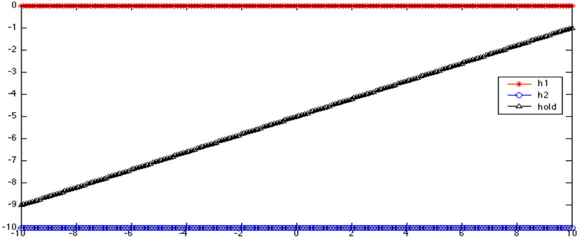
**Physical parameters for the simulation.** :

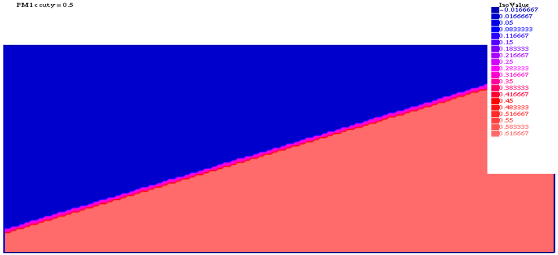
$\Omega = [-10, 10] \times [-10, 0]$ //2D domain,
$\Omega \times [-10, 0] = [-10, 10] \times [-10, 0] \times [-10, 0]$ //3D domain,
$\phi = 0.3, \quad K = \kappa = 39.024 m/day, \quad \delta = 0.1 m,$
$qs = 0.04 * (-9. <= x <= -7.) * (-6. <= y <= -4.) * (-2 <= z <= 0), \quad q_i = 0,$
$Q_f = 0.04 * (-9. <= x <= -7.) * (-6. <= y <= -4.), \quad Q_s = 0,$




Condition initiale c PM3

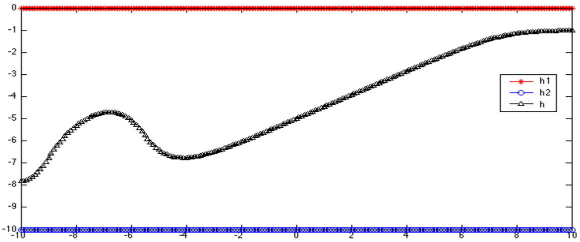
## initial interface

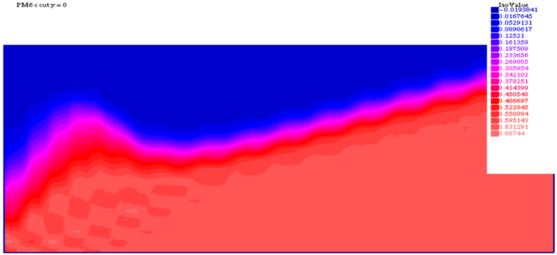


# Illustration

interface at N=1000




THANK YOU
FOR YOUR ATTENTION

Find $(p_h^n, u_h^n) \in W_h \times V_h$ such that :

$$\int_\Omega \phi \frac{p_h^n - p_h^{n-1}}{dt} \; w_h \; dX + \int_\Omega w_h \; \nabla \cdot u_h^n \; dX + \int_\Omega u_h^n \; v_h \; dX$$
$$= \int_\Omega (q_i - q_s) \; w_h \; dX + \int_\Omega \kappa p_h^n \; \nabla \cdot v_h \; dX, \qquad (7)$$

and the second problem is to find $(c_h^n, u_{ch}^n) \in W_h \times V_h$ such that :

$$\int_\Omega \phi \frac{c_h^n - c_h^{n-1}}{dt} \; w_h \; dX - \int_\Omega w_h \; c_h^n \nabla \cdot u_h^n \; dX + \int_\Omega w_h \; \nabla \cdot u_{ch}^n \; dX$$
$$+ \int_\Omega u_{ch} \; v_h \; dX - \int_\Omega u_h^n c_h^n \; v_h \; dX$$
$$= \int_\Omega q_i (1 - c_h^n) \; w_h \; dX + \int_\Omega \phi \mathcal{D} c_h^n \; \nabla \cdot v_h \; dX. \qquad (8)$$

pour tout $(w_h, v_h) \in W_h \times V_h$.