Mixed sharp-diffuse approach for seawater modeling

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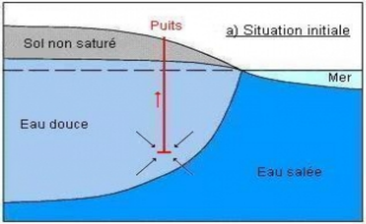

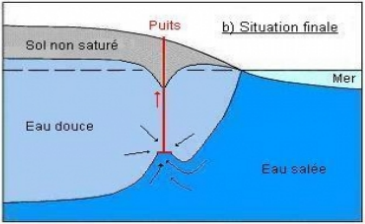

FIGURE : Aquifer without depression - Aquifer with depression

In fact there exist hydraulic exchanges between fresh groundwater and seawater.

They are slow in "natural conditions" (**Ghyben–Herzberg approximation**).

Goal

To obtain a efficient and accurate models to simulate the displacement of fresh and salt water fronts in coastal aquifer for the optimal exploitation of groundwater.

1. **Hidden diffuse interfaces**

   Fresh and salt water are two miscible fluids.

   Disadvantage : The approach is very heavy from theoretical and numerical points of view.

2. **Hidden sharp interfaces**

   Fresh and salt water are two immiscible fluids.

   Disadvantage : The explicit tracking of the interfaces remains unworkable to implement without further assumptions.

3. **Abrupt interfaces**

   The two fluids are immiscible and between the fresh and the salt area there are no mass transfert occurs.

   Advantage : This approach gives informations concerning the movement of the saltwater front.

   Disadvantage : But it doesn't describe the behavior of the real transition zone.

We propose a **mixed approach** between **abrupt interface** and **diffuse interface**.

Advantage :

- Adopt the (numerical) simplicity of a sharp interface approach,
- Refine the mathematical analysis of the free interface,
- Take account of the existence of the transition zone.

Two important assumptions to remember :

- No transfert of mass between fresh and salt water :
  ⊕ the appearance of the free interface,
  ⊕ an upscaling procedure reduces the dimension of the problem from 3D to 2D.

But

- the existence of a diffuse interface between fresh and salt :
  ⊕ a phase field approach (Allen-Cahn type model in fluid-fluid context).

Remark

The same assumptions is applied to the area of desaturation.

The derivation of the model 3D is based on the coupling of

- mass conservation principle : $\partial_t(\phi\rho) + \nabla\cdot(\rho q) = \rho Q,$
- Darcy's law : $q = -K\nabla\Phi - \frac{k}{\mu}(\rho - \rho_0)g\nabla z,$

written for freshwater and saltwater.

- **Dupuit approximation :** The averaged mass conservation law for the fresh/salt water

$$S_f B_f \partial_t \tilde{\Phi}_f = \nabla'\cdot(B_f \tilde{K}'_f \nabla'\tilde{\Phi}_f) - q_{f|z=h_1^-}\cdot\nabla(z - h_1^-) + q_{f|z=h^-}\cdot\nabla(z - h^-) + B_f\tilde{Q}_f, \quad (1)$$

$$S_s B_s \partial_t \tilde{\Phi}_s = \nabla'\cdot(B_s \tilde{K}'_s \nabla'\tilde{\Phi}_s) + q_{s|z=h_2}\cdot\nabla(z - h_2) - q_{s|z=h^-}\cdot\nabla(z - h^-) + B_s\tilde{Q}_s, \quad (2)$$

- **Continuity of hydraulic head and pression :**

$$\tilde{\Phi}_f = \frac{P_a}{\rho_f g} + h_1^- - h_{ref}. \quad (3)$$

$$(1+\alpha)\,\tilde{\Phi}_s = \frac{P_a}{\rho_f g} + h_1^- + \alpha h^- - (1+\alpha)h_{ref}, \quad \alpha = \frac{\rho_s}{\rho_f} - 1. \quad (4)$$

▶ **Fluxes across the interfaces** :

This part is fundamental because we re-include now existence of miscible zones,

• between fresh and salt water, the transition zone characterized by this thickness δ_h.

We introduce an order parameter F_h (the phase field)

$$F_h = \begin{cases} 0 & \text{in freshwater} \\ c_s/2 & \text{on sharp interface} \\ c_s & \text{in saltwater} \end{cases}$$

Set $\left\{(x_1, x_2, z)\, s.t.\, F_h(x_1, x_2, z, t) = c_s/2\right\}$ represents the sharp interface at time t.

F_h satisfies an equation of Allen-Cahn type with three points of stability :

$$\partial_t F_h + \vec{v} \cdot \nabla F_h - \delta_h \Delta' F_h + \frac{F_h(F_h - c_s/2)(F_h - c_s)}{\delta_h} = 0,$$

Using the implicite function result and including the result in the projection of Allen–Cahn equation for $F_h = c_s/2$, we get :

$$\partial_z F_h\big(-\partial_t h^- + \vec{v} \cdot \nabla(z - h^-) + \delta_h \Delta' h^-\big) + \delta_h \nabla' h^- \cdot \nabla' \partial_z F_h + \delta_h |\nabla' h^-|^2 \partial^2_{zz} F_h = 0. \quad (5)$$

The two last terms of the lefthand side of (5) may be neglected.

- $\delta_h \ll 1$ (the thickness of the transition zone is very small),
- $|\nabla' h^-| \ll 1$ (Dupuit's work 1863).

For the same reason, function F_h heuristically behaves like a step function in the vertical direction and $\partial_z F_h \neq 0$. The latter equation thus gives :

$$-\partial_t h^- + \vec{v} \cdot \nabla(z - h^-) + \delta_h \Delta' h^- = 0. \tag{6}$$

There is no mass transfer across the interface between fresh and salt water :

$$\left(\frac{q_{f|z=h^-}}{\phi} - \vec{v}\right) \cdot \vec{n} = \left(\frac{q_{s|z=h^-}}{\phi} - \vec{v}\right) \cdot \vec{n} = 0. \tag{7}$$

Combining (6) et (7), we obtain :

$$q_{f|z=h^-} \cdot \nabla(z - h^-) = \phi\big(\mathcal{X}_0(-h)\partial_t h - \delta_h \nabla' \cdot (\mathcal{X}_0(-h)\nabla' h)\big) \tag{8}$$

Relation (8) is a regularized Stefan type boundary condition.

We set $T_s(u) = h_2 - u$, $T_f(u) = u$ for $u \in (0, h_2)$ and we get when $S_f \ll 1$:

$$
(\mathcal{M}_2) \quad \begin{cases}
\phi\partial_t h - \nabla' \cdot \big(K\, T_s(h)\nabla' h\big) - \nabla' \cdot \big(\delta\phi\mathcal{X}_0(h_1)\nabla' h\big) \\
\qquad -\nabla' \cdot \big(K\, T_s(h)\nabla' h_1\big) - \phi q_{Ls}(x, h_1, h) = -\tilde{Q}_s\, T_s(h), \\
\phi\partial_t h_1 - \nabla' \cdot \Big(K\mathcal{X}_0(h_1)\big(T_f(h - h_1) + T_s(h)\big)\nabla' h_1\Big) \\
\qquad -\nabla' \cdot \big(\delta\phi\mathcal{X}_0(h_1)\nabla' h_1\big) - \nabla' \cdot \big(K\, T_s(h)\mathcal{X}_0(h_1)\nabla' h\big) \\
\qquad -\phi q_{Lf}(x, h_1, h) - \phi q_{Ls}(x, h_1, h) = -\tilde{Q}_f\, T_f(h - h_1) - \tilde{Q}_s\, T_s(h).
\end{cases}
$$

$$
h = h_D, \quad h_1 = h_{1,D} \quad \text{dans } \Gamma \times (0, T), \tag{9}
$$

$$
h(0, x) = h_0(x), \quad h_1(0, x) = h_{1,0}(x) \quad \text{dans } \Omega, \tag{10}
$$

We consider an open bounded domain Ω of $\mathbb{R}^2$, The boundary Γ, assumed $\mathcal{C}^1$.

The time interval of interest is $(0, T)$, $T > 0$, and we set $\Omega_T = (0, T) \times \Omega$.

$$
W(0, T) := \{\omega \in L^2(0, T; V),\ \partial_t\omega \in L^2(0, T; V')\},
$$

the embedding $W(0, T) \subset L^2(0, T; H)$ is compact.

Théorème 1 :

Assume a low spatial heterogeneity for the hydraulic conductivity tensor :

$$K_- \leq K_+ \leq 2K_-.$$

Then for any $T > 0$, problem $(\mathcal{M}_2)$-(10) admits a weak solution (h, h_1) satisfying

1. $(h - h_D, h_1 - h_{1,D}) \in W(0,T) \times W(0,T)$.
2. $0 \leq h_1(t,x) \leq h(t,x) \leq h_2$ for a.e. $x \in \Omega$ and for any $t \in (0,T)$.

The steps of the proof with the fixed point strategy

- We **regularize** the discontinuous function $\mathcal{X}_0$ (ϵ) and we introduce a **penalization based on the propagation of the front of salt water**.
- **Schauder's theorem** allows to affirm the existence of a solution to the problem regularized and penalized.
- The **maximum principle** coupled on h and h_1 is established on the regularized and penalized problem.
- We fight with estimates **to remove** the regularization and penalization.